Analytical calculation of the self-force on a part of a current loop using standard electrodynamics

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The self-force experienced by a semicircular conducting loop of circular cross section is evaluated analytically using the Lorentz force expression and shown to agree with the result of the partially analytical and partially numerical calculations quoted in the paper of Cavalleri *et al.* [Phys. Rev. E **58**, 2505 (1998)].

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In 1998 Cavalleri *et al.* published a paper [1] reporting the results of an experiment made by them and regarding the measurement of the self-force on a part of a circuit and due to the whole circuit. The force calculated by standard electrodynamics was found in agreement with the experimental values (within the experimental errors). They also pointed out experimental and theoretical errors made by other authors [2,3] who claimed disagreement between their experiments and the standard theory. The point is that not only is the experiment of a (usually weak) self-force delicate, but also the theoretical calculation is difficult since it concerns a sixfold integral whose integrand presents divergences. In order to calculate the self-force, one uses integrals over a distributed continuous force density, thus having a divergent integrand. That is why a Monte Carlo calculation is not convenient and practically does not converge. Consequently, Cavalleri *et al.* [1], after discussing other methods, succeeded in performing the analytical integration of one of the six integrals in cascade and then imitated the average distribution of the electrons (producing the current in the considered wire) that are equally spaced. They therefore divided the ranges of the other five variables into equal parts and calculated the force on each point excluding the action of the considered point on itself. For each number N of points for the main variable ϑ, they calculated numerically the fivefold integral I. The plot of I vs N, shown in their Fig. 11, does not reach an asymptotic value even for $N=10^3$ and for a total number of the five variables, $\sim 4\times 10^{11}$, requiring 1 month at full time of a PC for the last point. They therefore interpolated I vs N in order to find the asymptotic value. Since some uncertainty can arise in that procedure and in a reply to the comment made by Assis [4], Cavalleri and Tonni [5] have pointed out that an analytical solution of the sixfold integral would be worthwhile. In this paper we propose an analytical solution to the paper of Cavalleri *et al.* [1].

The results on the measurement of the self-force on a part of a closed circuit for a rectangular π frame is available in the literature [6,7]. For a rectangular π frame with rectangular cross section, the principal value of the diverging integral exists [8]. For the radius of the cross section tending to zero, the conductors of the π frame are assumed as line charges and the upward self-force is shown analytically to be a sum of infinite and finite values [6]. Moreover, for a π frame with finite radius of cross section the self-force is shown analytically [9] to be a sum of infinite and finite values where the finite value agrees with the experimental result [7]. The slow increase in the value of the self-force with the decrease in the value of the radius of the cross section is also correctly brought out [9]. In the present work, the analytical result for a circular cross section and axis shaped as a semicircular arc is obtained and shown to be a sum of infinite and finite values. The finite part agrees well with the result of [1].

The Lorentz force between two current-carrying conductors is small, and hence the response is usually not considered. As a consequence of the Lorentz force, the electrons in the conductor acquire a velocity, and if the force is not constant, a deviation in the uniform electron concentration arises. Subsequently a reaction force of Coulomb nature comes into play. However, if the force itself is small, the reaction is a second-order effect and can be neglected. In the case of a semicircular current-carrying loop, the self-force which is position dependent becomes the sum of infinite and finite values. For the infinite part, the consequence of the reaction force needs to be considered within an infinitesimal amount of time. It is physically equivalent to the redistribution of electrons when numerous point charges are kept in a free electron gas and the redistribution of electrons takes place within an infinitesimal amount of time. A random motion of electrons is associated with the redistribution process, and the net displacement of electrons along any direction is zero. Hence it would not contribute to the experimental measurement, which involves the displacement of electrons along the x direction for the present case. If the finite part of the force is large, then the calculation of the reaction force is difficult. In this paper, to overcome this difficulty, two equations for the force F_{Lx} are generated and solved. The uniqueness of the solution is also justified.

Considering two moving electrons with velocities $\vec{v}_1$ and $\vec{v}_2$, the Lorentz force experienced by electron 2 is written as

$$\vec{F} = -\frac{e\vec{v}_2\times(\vec{v}_1\times\vec{E})}{c^2}, \tag{1}$$

where $\vec{E}$ refers to the electric field. The x, y and z components of $\vec{E}$, E_1, E_2, and E_3 can be written as

$$E_1 = -\frac{e(x_2-x_1)}{r_{21}^3},$$

$$E_2 = -\frac{e(y_2-y_1)}{r_{21}^3},$$

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$$E_3 = -\frac{e(z_2 - z_1)}{r_{21}^3},$$

$$r_{21} = [\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\phi_1 - \phi_2) + (z_2 - z_1)^2]^{1/2}. \quad (2)$$

The total force experienced $\vec{F_L}$ by conductor 2 is obtained by summing $\vec{F}$ over the coordinates of electrons 1 and 2 and the summation can be replaced by an integration. The x and y coordinates of electrons 1 and 2 in the semicircular conducting loop of Fig. 1 can be written as

$$x_1 = \rho_1 \cos \phi_1,$$

$$y_1 = \rho_1 \sin \phi_1,$$

$$x_2 = \rho_2 \cos \phi_2,$$

$$y_2 = \rho_2 \sin \phi_2. \quad (3)$$

Here $14.3\ \text{cm} \leqslant \rho_1 \leqslant 14.8\ \text{cm}$ and $14.3\ \text{cm} \leqslant \rho_2 \leqslant 14.8\ \text{cm}$ [1]. The maximum value of z_1 and z_2 at each ρ_1 and ρ_2 can be written as

$$z_{1f} = \sqrt{\rho_0^2 - (\rho_1 - \rho_c)^2} = \rho_0\left(1 - \frac{(\rho_1 - \rho_c)^2}{\rho_0^2} - \cdots\right),$$

$$z_{2f} = \rho_0\left(1 - \frac{(\rho_2 - \rho_c)^2}{\rho_0^2} - \cdots\right), \quad (4)$$

where $\rho_c = 14.55$ cm. The radius of the cross section of the wire ρ_0 is 0.25 cm.

The x component of the Lorentz force can be written as

$$F_{Lx} = -\frac{e}{c^2}\sum (E_2 v_{1x} v_{2y} - E_1 v_{1y} v_{2y}), \quad (5)$$

where $v_{2x} = \frac{-i\sin(\phi_2)}{neA}$ and $v_{2y} = \frac{i\cos(\phi_2)}{neA}$. The area of cross section $A = \pi(0.25)^2$. Here n and i are the electron concentration and current in the loop.

The force experienced by the electrons in the loop can be evaluated by replacing the summation in Eq. (5) as an integration:

$$F_{Lx} = \frac{n^2e^2}{c^2}\int \rho_1 d\rho_1 \int \rho_2 d\rho_2 \int dz_2 \int dz_1 \times \int d\phi_2 \int (E_2 v_{1x} v_{2y} - E_1 v_{1y} v_{2y}) d\phi_1, \quad (6)$$

with the limits of integration as below. Here $-\frac{\pi}{2} \leqslant \phi_1 \leqslant \frac{\pi}{2} - \frac{\pi}{2} \leqslant \phi_2 \leqslant \frac{\pi}{2}$, $14.3\ \text{cm} \leqslant \rho_1 \leqslant 14.8\ \text{cm}$, $14.3\ \text{cm} \leqslant \rho_2 \leqslant 14.8\ \text{cm}$, $-z_{1f} \leqslant z_1 \leqslant z_{1f}$, and $-z_{2f} \leqslant z_2 \leqslant z_{2f}$. The contribution from the term $E_1 v_{1y} v_{2y}$ is zero as the orders of integration for the coordinates of electrons 1 and 2 can be interchanged. Hence

$$F_{Lx} = -\frac{i^2}{A^2c^2}\int \rho_2 d\rho_2 \int \rho_1 d\rho_1 \int dz_2 \int dz_1 \times \int \cos \phi_2 d\phi_2 \times \int \frac{[\rho_2 \sin \phi_2 - \rho_1 \sin \phi_1]\sin \phi_1}{r_{21}^3} d\phi_1. \quad (7)$$

The sixfold integrations are evaluated analytically as below.

r_{21}^3 can be expanded as a binomial series. In evaluating the $d\phi_1$ and $d\phi_2$ integrals, the contribution from $(\rho_2 \sin \phi_2 - \rho_1 \sin \phi_1)\sin \phi_1 \cos \phi_2$ and the terms of odd powers of $\cos(\phi_1 - \phi_2)$ in the binomial series become zero. Considering the terms of even powers of $\cos(\phi_1 - \phi_2)$ in the binomial series, one can redefine F_{Lx} as

$$F_{Lx} = \frac{\pi i^2}{2A^2c^2}\int \rho_1^2 d\rho_1 \int \rho_2 d\rho_2 \int \cos \phi_2 d\phi_2 \times \int\int \frac{dz_1 dz_2}{[\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2 + (z_2 - z_1)^2]^{3/2}}. \quad (8)$$

One can write

$$\int \frac{dz_1}{[\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2 + (z_2 - z_1)^2]^{3/2}} = \frac{z_1 - z_2}{\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2}\frac{1}{[\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2 + (z_2 - z_1)^2]^{1/2}}. \quad (9)$$

Hence, on performing the dz_2 integration and applying the limits for the dz_1 and dz_2 integrals, the right-hand side of Eq. (9) becomes

$$\left[\frac{-2}{\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2}\right] \times \{\sqrt{[\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2 + (z_{2f} - z_{1f})^2]} - \sqrt{[\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2 + (z_{2f} + z_{1f})^2]}\}.$$

Since $\frac{z_{1f}}{\rho_1}$ and $\frac{z_{2f}}{\rho_2}$ are very small, considering only the terms containing $(z_{1f}+z_{2f})^2$ and $(z_{1f}-z_{2f})^2$, F_{Lx} can be written as

$$F_{Lx} = \frac{2\pi i^2}{c^2A^2}\int\int \rho_1^2 \rho_2 z_{1f} z_{2f} d\rho_1 d\rho_2 \times \int \frac{\cos \phi_2}{[\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2]^{3/2}} d\phi_2. \quad (10)$$

After the $d\phi_2$ integration one gets

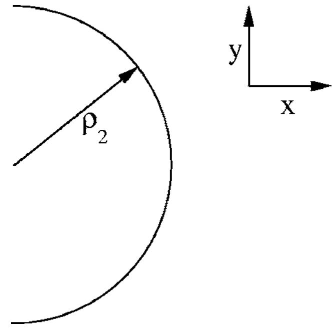

FIG. 1. Semicircular loop and axis convention.

$$F_{Lx} = \frac{4\pi i^2}{c^2 A^2} \int d\rho_2 \int_{14.3}^{\rho_2} \frac{\rho_1 z_{1f} z_{2f}}{(\rho_2 - \rho_1)} d\rho_1 . \tag{11}$$

Defining

$$\rho_1 - \rho_c = \rho_0 \sin \theta_1 ,$$

$$\rho_2 - \rho_c = \rho_0 \sin \theta_2 , \tag{12}$$

one can write

$$\frac{1}{\rho_2 - \rho_1} = \frac{1}{\rho_0 (\sin \theta_2 - \sin \theta_1)} . \tag{13}$$

Using Eq. (4) and the binomial series for $\frac{1}{\rho_2 - \rho_1}$, the $d\rho_1$ integral is evaluated. The limits of integration are from -1 to $\sin^{-1}\left(\frac{\rho_2 - \rho_c}{\rho_0}\right)$. The subsequent $d\theta_2$ integral becomes tedious to evaluate analytically.

Instead $[\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2]^{-3/2}$ can be written as a binomial series and the $d\phi_2$ integration performed. The terms having even powers of $\sin \phi_2$ alone contribute to the $d\phi_2$ integral from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Two equations for F_{Lx} are generated below and solved so that the finite part of F_{Lx} becomes small. One can write

$$\int \frac{\cos \phi_2}{[\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \sin \phi_2]^{3/2}} d\phi_2 = \frac{1}{(\rho_1^2 + \rho_2^2)^{3/2}} S_1 ,$$

where

$$S_1 = 2[1 + 0.625x^2 + 0.492x^4 + 0.418x^6 + 0.370x^8 + 0.33x^{10} + \cdots]$$

$$x = \frac{2\rho_1\rho_2}{\rho_1^2 + \rho_2^2} . \tag{14}$$

x is less than unity. However, it is very nearly unity for all values of ρ_1 and ρ_2. For $\rho_1 = 14.8$ cm and $\rho_2 = 14.3$ cm, $x = 0.9994$. Hence the series can also be written as

$$S_1 = 2[1 + 0.625 + 0.492 + 0.418 + 0.370 + 0.33 + \cdots] = 2\left[2.905 + 0.33 \frac{1}{(1 - 0.9x)} + S_2\right] ,$$

where the series S_2 is of the form

$$S_2 = 0.011x + 0.0197x^2 + 0.0285x^3 + \cdots . \tag{15}$$

S_2 can be rewritten as

$$S_2 = 0.011 \frac{1}{(1-x)^2} + S_3 ,$$

where

$$S_3 = -[0.0023 + 0.0045x + 0.0065x^2 + \cdots] = -2.2966 - \sum_{n=0}^{m} [0.2183 + n0.0117]x^{n+29} + 0.0117(m-30)x^{m-30} + \cdots + 0.0117mx^m + 0.2183(x^{m-30} + \cdots + x^m) \tag{16}$$

and m is ∞

Hence

$$S_1 = 5.8 + 2S' ,$$

where

$$S' = 0.33 \frac{1}{(1-0.9x)} + 0.011 \frac{1}{(1-x)^2} - 2.2966 - \sum_{n=0}^{m} [0.2183 + n0.0117]x^{n+29} + \sum_{n=m-30}^{m} [0.2183 + n0.0117]x^n . \tag{17}$$

Similarly one can represent S' by a combination of different terms as below:

$$S' = 0.33 \frac{1}{(1-0.91x)} + 0.008 \frac{1}{(1-x)^2} - 1.3638 - \sum_{n=0}^{m} (0.1358 + 0.0085n)x^{n+29} + \sum_{n=m-30}^{m} [0.1358 + n0.0085]x^n . \tag{18}$$

Using the standard relation for an algebric geometric progression [10],

$$\sum_{k=0}^{m-1} (a + kr)q^k = \frac{a - [a + (m-1)r]q^m}{(1-q)} + \frac{rq(1-q^{m-1})}{(1-q)^2} , \tag{19}$$

and Eqs. (17) and (18), S' is evaluated. As x is very nearly unity, the terms $\frac{1}{(1-0.9x)}$ and $\frac{1}{(1-.0.91x)}$ are practically constants independent of ρ_1 and ρ_2. Multiplying Eq. (18) by $\frac{0.2183}{0.1358}$ and subtracting Eq. (17), terms proportional to $\frac{1}{(1-x)}$ can be eliminated. Hence

$$0.607S' = 2.687 + \frac{0.0001}{(1-x)^2} + 0.0019 \sum_{n=m-30}^{m} nx^n . \tag{20}$$

Since $\frac{\rho_1^2\rho_2}{(\rho_1^2+\rho_2^2)^{3/2}}$ is nearly $2^{-3/2}$, using Eqs. (10) and (14) one can write

$$F_{Lx} = \frac{2\pi i^2}{c^2 A^2 2^{3/2}} \int S_1 z_{1f} z_{2f} d\rho_1 d\rho_2 . \tag{21}$$

The contribution from the terms involving $\frac{1}{(1-x)^2}$ is evaluated below, and the value of finite term is shown to be zero.

Substituting $\frac{1}{(1-x)^2}=\frac{(\rho_1^2+\rho_2^2)^4}{(\rho_2-\rho_1)^4}$ and defining $\rho_1-\rho_c=\rho_0 \sin\theta_1$, one gets

$$\int\int \frac{1}{(\rho_2-\rho_1)^4} z_{1f} z_{2f} d\rho_1 d\rho_2 = \int \sqrt{\rho_0^2-(\rho_2-\rho_c)^2} d\rho_2 \times \int \frac{\rho_0^2 \cos^2\theta_1}{[\rho_2-\rho_c-\rho_0 \sin\theta_1]^4} d\theta_1, \quad (22)$$

with the limits $-\frac{\pi}{2}\leqslant\theta_1\leqslant\sin^{-1}\left(\frac{\rho_2-\rho_c}{\rho_0}\right)$ and $14.3\text{cm}\leqslant\rho_2\leqslant 14.8$ cm. For $\rho_2\geqslant\rho_1$,

$$\frac{1}{[\rho_2-\rho_c-\rho_0 \sin\theta_1]^4}=\frac{1}{(\rho_2-\rho_c)^4}\left[1+\frac{4\rho_0 \sin\theta_1}{(\rho_2-\rho_c)}+\cdots\right]. \quad (23)$$

$\cos^2\theta_1$ can be written as $\frac{1+\cos 2\theta_1}{2}$. The contribution from the $\cos 2\theta_1$ term after evaluating the $d\theta_1$ integral in Eq. (22) either becomes zero or proportional to odd powers of $\frac{(\rho_2-\rho_c)}{\rho_0}$. As the limits of $\rho_2-\rho_c$ are from -0.25 to 0.25, the value of the integrals becomes zero. Similarly considering $\frac{1}{2}$ in $\cos^2\theta_1$, the contribution from the terms of odd powers of $\sin\theta_1$ on the right-hand side of Eq. (23) becomes zero. The terms of even powers of $\sin\theta_1$ in Eq. (23) after $d\theta_1$ integration reduce to integrals of the form

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_2}{\sin^{2n}\theta_2} d\theta_2,$$

where $\rho_2-\rho_c=\rho_0 \sin\theta_2$ and n is greater than unity. The principal value of the integral at $\theta_2=0$ is infinity. However, the value of the integral at the upper and lower limits is zero [10]. For ρ_1 greater than ρ_2, the order of $d\rho_1$ and $d\rho_2$ integrations can be interchanged and the same result holds good. Hence the value of finite term is zero and in Eq. (20) the term proportional to $\frac{1}{(1-x)^2}$ can be neglected.

Moreover, as m is infinity the terms involving $0.0117(m-30)x^{m-30}+\cdots+0.0117mx^m$ and $0.0085(m-30)x^{m-30}+\cdots+0.0085mx^m$ can be taken as infinity. Considering only the finite term, one gets

$$0.607S'=2.687. \quad (24)$$

Hence S' and S_1 become 4.42 and 14.6, respectively.

It may appear that S' can be evaluated directly using Eqs. (17) and (18), but it is less accurate as explained below. Using steps similar to Eqs. (22) and (23), the contribution from the $\frac{1}{(1-x)}$ term is proportional to the integral

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_2}{\sin^2\theta_2}\left[1+\frac{3}{2\sin^2\theta_2}+\frac{15}{8\sin^2\theta_2}+\frac{35}{16\sin^2\theta_2}\right] d\theta_2. \quad (25)$$

The principal value of the integral at $\theta_2=0$ is infinity. However, the value of the integral

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_2}{\sin^2\theta_2} d\theta_2$$

at the upper and lower limits is finite. One gets S' as approximately equal to 1000. As the force becomes a large finite value, the evaluation of the reaction force becomes less accurate. The uniqueness of the solution is justified below.

One can generate many equations similar to Eqs. (17) and (18) by changing the terms $\frac{1}{1-0.9x}$ and $\frac{1}{1-0.91x}$ and solving any two of them to obtain the value of S'. As an example two equations corresponding to $\frac{1}{1-0.8x},\frac{1}{1-0.9x}$ and $\frac{1}{1-0.88x},\frac{1}{1-0.91x}$ are solved and the values of S' are also obtained nearly as 4.4, thus justifying the uniqueness of the solutions.

Using Eq. (21) and substituting S_1 as 14.6, one can write

$$F_{Lx}=\frac{29.2\pi i^2}{c^2A^2 2^{3/2}}\int\int z_{1f}z_{2f}d\rho_1 d\rho_2. \quad (26)$$

Using Eqs. (4) with the limits $-\frac{\pi}{2}\leqslant\theta_1\leqslant\frac{\pi}{2}$, $-\frac{\pi}{2}\leqslant\theta_2\leqslant\frac{\pi}{2}$, and substituting $A=\pi(0.25)^2$, one gets

$$F_{Lx}=\frac{29.2(0.25)^4\pi i^2}{c^2A^2 2^{3/2}}\int\int \cos^2\theta_1 \cos^2\theta_2 d\theta_1 d\theta_2=\frac{8.1i^2}{c^2}. \quad (27)$$

One can also put forward an alternate method of solution based on elliptic integrals for the self-force on a semicircular conducting loop without using the simplification in Eq. (8) and obtain the finite part of the force as $\frac{9.2i^2}{c^2}$[9].

The results show agreement with the value $\frac{8.8i^2}{c^2}$ of [1].

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